A.3 Calculus

A.15. ***Integration by parts*** is a technique for simplifying integrals of the form

$$\int a(x)\,b(x)dx.$$

In particular,

$$\int f(x)\,g'(x)dx = f(x)\,g(x) - \int f'(x)\,g(x)dx. \tag{58}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$. Using the Substitution Rule, the integration by parts formula becomes

$$\int udv = uv - \int vdu \tag{59}$$

- The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate then the original. In other words, the goal of integration by parts is to go from an integral $\int udv$ that we dont see how to evaluate to an integral $\int vdu$ that we can evaluate.
- Note that when we calculate v from dv, we can use *any* of the antiderivative. In other words, we may put in $v + C$ instead of v in (59). Had we included this constant of integration C in (59), it would have eventually dropped out. This is always the case in integration by parts.

For definite integrals, the formula corresponding to (58) is

$$\int_a^b f(x)\,g'(x)dx = f(x)\,g(x)|_a^b - \int_a^b f'(x)\,g(x)dx. \tag{60}$$

The corresponding u and v notation is

$$\int_a^b udv = uv|_a^b - \int_a^b vdu \tag{61}$$

It is important to keep in mind that the variables u and v in this formula are functions of x and that the limits of integration in (61) are limits on the variable x. Sometimes it is helpful to emphasize this by writing (61) as

$$\int_{x=a}^{b} u dv = uv|_{x=a}^{b} - \int_{x=a}^{b} v du \tag{62}$$

Repeated application of integration by parts gives

$$\int f(x)\, g(x) dx = f(x)\, G_1(x) + \sum_{i=1}^{n-1} (-1)^i f^{(i)}(x)\, G_{i+1}(x) + (-1)^n \int f^{(n)}(x)\, G_n(x)\, dx \tag{63}$$

where $f^{(i)}(x) = \frac{d^i}{dx^i} f(x)$, $G_1(x) = \int g(x) dx$, and $G_{i+1}(x) = \int G_i(x) dx$.

A convenient method for organizing the computations into two columns is called ***tabular integration by parts*** shown in Figure 45 which can be used to derived (63).

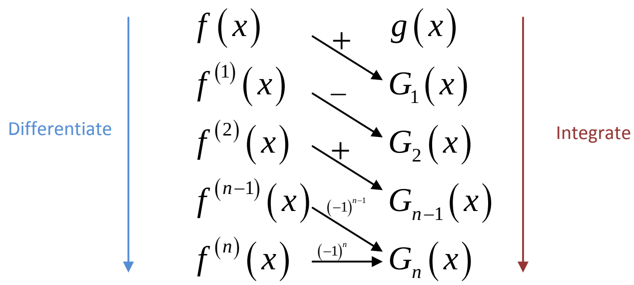

Figure 45: Integration by Parts

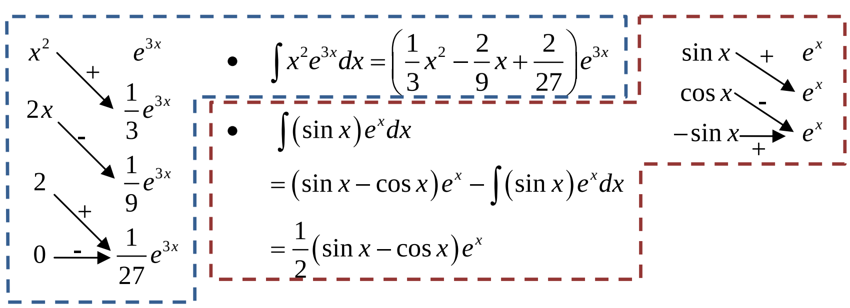

Figure 46: Examples of Integration by Parts using Figure 45.